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Multiple scattering of charged particles in irregular magnetic fields

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Abstract. Charged particle propagation in irregular magnetic fields is investigated. Multiple small angle scattering is considered in a model field configuration. The irregular magnetic field components are specified by their energy density and autocorrelation function. The mean deviation from ideal particle direction is given explicitly. A diffusion equation with variable diffusion coefficient is established, ordinary diffusion theory is found to be valid asymptotically. Results are applied for an estimation on the range of validity of the single-particle approach to the propagation of primary cosmic ray particles at extremely high energies.

1. Introduction

Application of the individual-trajectory approach to the problem of primary cosmic ray particle transfer at extremely high energies in given models of a magnetic field in our galaxy (Thielheim and Langhoff 1968, 1970, Karakula *et al* 1971, Osborne and Wolfendale 1973) is limited by the presence of field irregularities which give rise to multiple small angle scattering.

We want to estimate the mean change of particle direction for trajectories passing through a layer in which there exists an irregular magnetic field, the properties of which are specified by its energy density and its autocorrelation function.

The situation is different from problems which are being studied frequently by perturbation methods: we shall not use the restriction of random variations of the field strength to be small in comparison with the mean field strength. Particle energies considered here are extremely high. Therefore, the mean radius of curvature of particle trajectories remains large in comparison with the autocorrelation length of the irregular contribution to the field.

In order to make possible the direct and exact calculation of parameters related to the scattering of high energy particles, invariance properties are established for the model configuration of the magnetic field resulting in a sufficient number of constants of the motion.

2. Invariance properties and constants of the motion

In view of the physical conditions of interstellar plasma and the properties of high energy cosmic ray particles, the magnetic field may be considered to be stationary. The components of the field vector therefore, are time-independent functions of the coordinates

x, y, and z. Consequently particle energy is a constant of the motion:

$$\dot{x}^2+\dot{y}^2+\dot{z}^2 = c^2. \tag{1}$$

In addition, translational invariance of the magnetic field is introduced with respect to two coordinates, the x and y coordinates. The components of the field vector then depend on the z coordinate only. Consequently, the x and y components of generalized particle momentum also become constants of the motion.

Since the divergence of the magnetic field vector is known to vanish the z component of the field vector is therefore constant. Referring to the condition mentioned previously, according to which the mean radius of curvature is large in comparison with the autocorrelation length, this arbitrary constant is proposed to be zero.

As an illustration of the field topography one may imagine that within any given plane which is vertical to the z direction, the field lines are parallel and equidistant.

Eventually, as a consequence of translational invariance the components of particle velocity may be given explicitly as functions of the z coordinate:

$$\dot{x}(z) = \dot{x}_0 - eZcE^{-1}\int_0^z \mathrm{d}z'\, H_y(z'), \tag{2}$$

$$\dot{y}(z) = \dot{y}_0 + eZcE^{-1}\int_0^z \mathrm{d}z'\, H_x(z'), \tag{3}$$

where eZ is the charge and E is the energy of the particle. The changes in the x and y components of particle velocity are independent from their initial values.

3. Particle trajectories in given field configurations

The two remaining non-trivial field components may be expressed by Fourier series within a range $0 \leqslant z \leqslant L$:

$$H_x(z) = \sum_{k=1}^{\infty} [a_k \cos(2\pi kz/L) + b_k \sin(2\pi kz/L)], \tag{4}$$

$$H_y(z) = \sum_{k=1}^{\infty} [p_k \cos(2\pi kz/L) + q_k \sin(2\pi kz/L)], \tag{5}$$

where it is proposed that the constant terms vanish. The x and y components of particle velocity then also result in terms of Fourier series:

$$\dot{x}(z) = \dot{x}_0 - (LeZc/2\pi E) \sum_{k=1}^{\infty} [p_k \sin(2\pi kz/L) + 2q_k \sin^2(\pi kz/L)]k^{-1}, \tag{6}$$

$$\dot{y}(z) = \dot{y}_0 + (LeZc/2\pi E) \sum_{k=1}^{\infty} [a_k \sin(2\pi kz/L) + 2b_k \sin^2(\pi kz/L)]k^{-1}. \tag{7}$$

4. Random field components and ensembles of particle trajectories

The coefficients a_k, b_k, p_k, and q_k are now considered to be independent random variables, the variances of which depend on k only:

$$\langle a_k^2\rangle = \langle b_k^2\rangle = \langle p_k^2\rangle = \langle q_k^2\rangle = s_k^2. \tag{8}$$

Consequently, the x and y components of the magnetic field vector as well as of particle velocity become stationary random functions of the z coordinate. The angular brackets are used to indicate averaging either over the z coordinate or else, by the ergodic hypothesis, over an ensemble of realizations of the magnetic field.

The autocorrelation function which, of course, is the same for the two field components is related to the spectrum of irregularities:

$$\langle H_x(z)H_x(z+\zeta)\rangle = \sum_{k=1}^{\infty} s_k^2 \cos(2\pi k\zeta/L). \tag{9}$$

After transition from Fourier series to Fourier integrals ($L \to \infty$, $k/L \to \kappa$, $1/L \to \mathrm{d}\kappa$, $Ls_k^2 \to w(\kappa)$) this relation is obtained in the form:

$$\langle H_x(z)H_x(z+\zeta)\rangle = \int_0^{\infty} \mathrm{d}\kappa\, w(\kappa) \cos(2\pi\kappa\zeta). \tag{10}$$

The mean energy density of the irregular field may also be expressed by the spectrum of irregularities

$$\langle \rho \rangle = \frac{1}{4\pi}\int_0^{\infty} \mathrm{d}\kappa\, w(\kappa). \tag{11}$$

Thus, for given initial values of particle position and velocity, instead of perfectly determined trajectories, statistical ensembles of trajectories result. The mean change of the square of the sine of the pitch angle (that is, the angle between particle momentum and the z direction) is found explicitly as a function of z:

$$\langle \Delta \sin^2 \theta \rangle = 2e^2 Z^2 E^{-2} \int_0^z \mathrm{d}z_1 \int_0^z \mathrm{d}z_2 \langle H_x(z_1)H_x(z_2)\rangle. \tag{12}$$

5. Form of the autocorrelation function

We suggest the autocorrelation function decays exponentially

$$\langle H_x(z)H_x(z+\zeta)\rangle = 4\pi\langle \rho \rangle \exp(-2\pi|\zeta|/l_0) \tag{13}$$

with the autocorrelation length l_0 related to the cloudy structure of interstellar hydrogen. The same form of the autocorrelation function has been found in other phenomena of turbulent continuum mechanics. The resulting spectrum of irregularities is:

$$w(\kappa) = \frac{8\langle \rho \rangle l_0}{1+\kappa^2 l_0^2}. \tag{14}$$

It should be mentioned that the first derivative of the autocorrelation function (13) is not defined at $\zeta = 0$. In view of the symmetry of the autocorrelation function, this value should be re-defined as zero. Nevertheless, the first derivative of the autocorrelation function remains discontinuous at $\zeta = 0$. Correspondingly, it is related to a magnetic field component $H_x(z)$ represented by a non-continuously differentiable function of z. This unpleasant situation may be overcome by means of an adequate auxiliary function, eg

$$\langle H_x(z)H_x(z+\zeta)\rangle_\epsilon = 4\pi\langle \rho \rangle \left(\frac{l_0}{l_0-\epsilon}\exp(-2\pi|\zeta|/l_0) - \frac{\epsilon}{l_0-\epsilon}\exp(-2\pi|\zeta|/\epsilon)\right) \tag{15}$$

which is continuously differentiable at $\zeta = 0$ and reduces to (13) for $\epsilon \to 0$.

6. Mean deviations from ideal trajectories

Insertion of (13) into (12) gives the mean change of the square of the sine of the pitch angle as a function of the z coordinate:

$$\langle \Delta \sin^2 \theta \rangle = 4\pi^{-1} e^2 Z^2 \langle \rho \rangle l_0^2 E^{-2} [2\pi z/l_0 - 1 + \exp(-2\pi z/l_0)]. \tag{16}$$

Limiting expressions are

$$\langle \Delta \sin^2 \theta \rangle = 8\pi e^2 Z^2 \langle \rho \rangle E^{-2} z^2 \qquad \text{for } z \ll l_0, \tag{17}$$

where the proportionality to z^2 demonstrates the prevalence of causality in particle transfer, and

$$\langle \Delta \sin^2 \theta \rangle = 8 e^2 Z^2 \langle \rho \rangle l_0 E^{-2} z \qquad \text{for } z \gg l_0, \tag{18}$$

where the proportionality to z indicates that particles propagate essentially at random in a diffusion-type process. We are, of course, interested in the latter expression.

7. Asymptotic diffusion theory

Although the problem presented at the beginning of this paper is essentially solved by (18) we wish, in the course of a more detailed discussion, to consider the combined distribution function $f(\dot{x}, \dot{y}|z)$ in the $\dot{x}$, $\dot{y}$ plane at a given position z. We also wish to verify under which conditions this combined distribution function reduces to a solution of a diffusion equation.

Due to the independence of these two random variables $\dot{x}$ and $\dot{y}$,

$$f(\dot{x}, \dot{y}|z) = f(\dot{x}|z) f(\dot{y}|z), \tag{19}$$

where, according to the central limit theorem,

$$f(\dot{x}|z) = [2\pi \langle (\Delta \dot{x})^2 \rangle]^{-1/2} \exp[-(\Delta \dot{x})^2 / 2\langle (\Delta \dot{x})^2 \rangle], \tag{20}$$

$$f(\dot{y}|z) = [2\pi \langle (\Delta \dot{y})^2 \rangle]^{-1/2} \exp[-(\Delta \dot{y})^2 / 2\langle (\Delta \dot{y})^2 \rangle]. \tag{21}$$

The combined distribution function $f(\dot{x}, \dot{y}|z)$ is thus found to obey the following partial differential equation:

$$\left(\frac{\partial^2}{\partial \dot{x}^2} + \frac{\partial^2}{\partial \dot{y}^2} \right) f(\dot{x}, \dot{y}|z) = \frac{1}{D(z)} \frac{\partial}{\partial z} f(\dot{x}, \dot{y}|z), \tag{22}$$

which may be understood as a diffusion equation with variable diffusion coefficient:

$$D(z) = e^2 Z^2 c^2 E^{-2} \int_0^z \mathrm{d}z' \langle H_x(z) H_x(z') \rangle. \tag{23}$$

'Ordinary' diffusion theory enters asymptotically when z is large in comparison with the autocorrelation length. The diffusion coefficient then becomes constant,

$$D = \tfrac{1}{4} e^2 Z^2 c^2 w(0) E^{-2}, \tag{24}$$

with (13) one obtains:

$$D = 2 e^2 Z^2 c^2 \langle \rho \rangle l_0 E^{-2}. \tag{25}$$

8. Range of applicability of the individual-trajectory approach

Introducing adequate units ($eZ = 4{\cdot}082 \times 10^{-10}\, g^{1/2}\, cm^{3/2}\, s^{-1}$ for protons, $1\, eV = 1{\cdot}602 \times 10^{-12}\, erg$, $1\, pc = 3{\cdot}085 \times 10^{18}\, cm$) the result given in (18) obtains the form:

$$\langle \Delta \sin^2 \theta \rangle = 4{\cdot}94 \times 10^{42} \langle \rho \rangle l_0 z E^{-2}, \tag{26}$$

($\langle \rho \rangle$ in erg cm^{-3}, l_0 and z in pc, E in eV).

This relation may be used for an estimation on the range of particle energy in which the individual trajectory approach may be applied. The mean interstellar magnetic field strength is about 3 μG (Mills 1971) corresponding to an energy density of $3{\cdot}5 \times 10^{-13}$ erg cm^{-3}. One may guess (more or less arbitrarily) that about 20 % of this energy density, ie $0{\cdot}7 \times 10^{-13}$ erg cm^{-3} may be attributed to the turbulent field components. The typical diameter of interstellar clouds is of the order of 10 pc. The thickness of the layer which typically has to be traversed by protons of extremely high energy is about the double scale height, 300 pc. Under these conditions deviations from the ideal particle trajectories are found to be of the order of 2° at $E = 10^{18}$ eV.

References

Karakula S, Osborne J L, Roberts E and Thaczyk W 1971 *Proc. 12th Int. Conf. on Cosmic Rays, Hobart* vol 1 (Hobart: University of Tasmania) pp 310–5

Mills B Y 1971 *Proc. 12th Int. Conf. on Cosmic Rays, Hobart* (Hobart: University of Tasmania)

Osborne J L and Wolfendale A W 1973 *Proc. 13th Int. Conf. on Cosmic Rays, Denver* vol 1 (Denver: University of Denver) pp 610–5

Thielheim K O 1972 *Proc. 'Schule fur Weltraumphysik'* Titi-See (West Germany)

Thielheim K O and Langhoff W 1968 *J. Phys. A: Gen. Phys.* **1** 694–703

—— 1970 *Acta Phys. Acad. Sci. Hung.* **29** *Suppl.* 1, 549–55